Chapter 2

Earth's Gravity Field

The knowledge of earth's gravity field is crucial to understand how to determine geoid. The gravitation and centrifugal force are the basic theory of gravity. Before dealing with earth's gravity field, the knowledge of three dimensional geodetic system is also given.

2.1 Three Dimensional Geodetic Coordinate System

In geodesy, there are two global coordinate system currently used. They are earth-fixed rectangular and geodetic coordinate system. Earth-fixed rectangular coordinate system has three arguments X, Y and Z while geodetic arguments are φ (geodetic latitude), λ (geodetic longitude) and h (geodetic height). Earth-fixed rectangular coordinate system cannot be divided into horizontal and vertical components, which make it impossible to make interpretation about height. To make the interpretation about height easier, earth-fixed rectangular coordinate system is usually converted into geodetic coordinate system. Geodetic coordinates are related to earth-fixed rectangular coordinates by the equations (see Figure 2.1)

$$
\begin{aligned}
X &= (R_N + h)\cos\lambda\cos\varphi \\
Y &= (R_N + h)\cos\lambda\sin\varphi \\
Z &= (R_N(1 - e^2) + h)\sin\lambda,
\end{aligned}
\tag{2.1}
$$

where

$$N = \frac{a}{\sqrt{(1 - e^2 \sin^2 \varphi)}} \tag{2.2}$$

is ellipsoid main vertical curve radius and

$$e = \frac{(a^2 - b^2)}{a^2} \tag{2.3}$$

is ellipsoid first eccentricity. Ellipsoid major axes a and ellipsoid minor axes b are specified by the reference ellipsoid used. Geodetic coordinates are divided into two components, horizontal (φ, λ) and vertical(h). Geodetic coordinates can be easily determined by the use of GPS (Global Positioning System) and or other satellite positioning system, such as GLONASS, Galileo and COMPASS, which together are called GNSS (Global Navigation Satellite System).

2.2 Gravity and Gravity Potential

Gravity is defined as the resultant of gravitational force and centrifugal force (see Figure 2.2). Gravitational force is caused by mass attraction and the centrifugal force is caused by the earth's rotation.

According to Newton law of gravitation two points with masses m_1 and m_2 separated by a distance l, attract each other with a force

$$F = G\frac{m_1 m_2}{l^2}, \tag{2.4}$$

where G is the Newtonian gravitational constant and has the value

$$G = 6.672X10^{-11} m^3 kg^{-1} \sec^{-2}. \tag{2.5}$$

Gravitation force F can be formulated in the earth-fixed rectangular coordinate system as $F(X, Y, Z)$. Force vector F is the gradient vector of the scalar function V which is called the potential of gravitation

$$F(X, Y, Z) = grad\, V \tag{2.6}$$

and

$$V = \frac{Gm}{l}. \tag{2.7}$$

It is of basic importance that the three components of the vector F can be replaced by single scalar function V. It is much easier to deal with the potential than with the three components of the force especially when we consider the attraction of point masses or solid bodies, as we do in geodesy (Heiskanen &

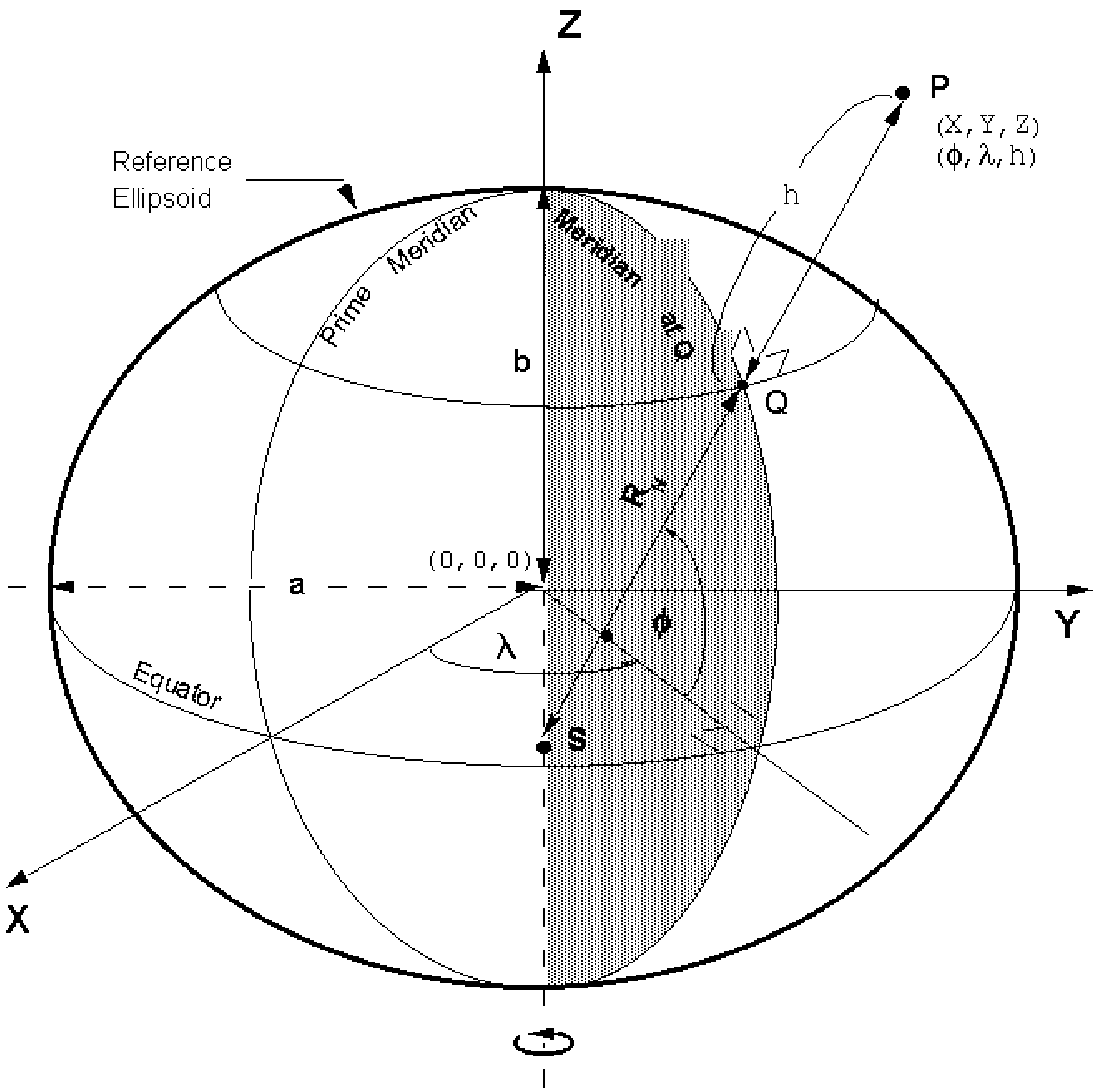

Figure 2.1: Geodetic and geocentric coordinate system relationship (Figure from http://geoengine.nga.mi)

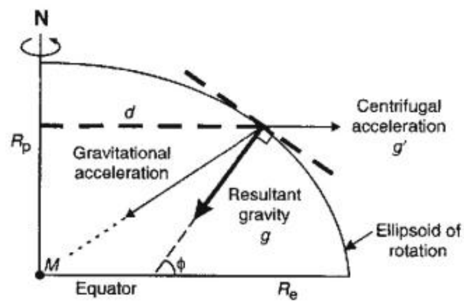

Figure 2.2: Gravity force as the resultant of gravitation force and centrifugal force (Figure from http://pkukmweb.ukm)

Moritz, 1967). The function V is simply the sum of the contributions of the respective masses. The potential of n point masses is

$$V = \frac{Gm_1}{l_1} + \frac{Gm_2}{l_2} + \cdots + \frac{Gm_n}{l_n} = G\sum_{i=1}^{n}\frac{m_i}{l_i}, \tag{2.8}$$

then the potential of earth is given as

$$V = G \iiint\limits_{Earth} \frac{dm}{l}, \tag{2.9}$$

where dm is the mass element. Because density

$$\rho = \frac{dm}{dv}, \tag{2.10}$$

then (2.9) become

$$V = G \iiint\limits_{Earth} \frac{\rho}{l} dv. \tag{2.11}$$

The potential of the centrifugal force Φ, can be formulated in the earth-fixed rectangular coordinate system as

$$\Phi = \frac{1}{2}\omega^2 \left(X^2 + Y^2\right), \tag{2.12}$$

where polar motion is neglected, ω is the mean angular velocity of the earth's rotation.

Gravity potential W as the resultant of gravitational potential and centrifugal potential can be formulated in the earth-fixed rectangular coordinate as

$$W(X, Y Z) = V(X, Y, Z) + \Phi(X, Y, Z). \tag{2.13}$$

Inserting (2.11) and (2.12) into (2.13) give

$$W = G \iiint\limits_{Earth} \frac{\rho}{l} dv + \frac{1}{2}\omega^2 \left(X^2 + Y^2\right). \tag{2.14}$$

The gradient of W produces the gravity vector $\overline{g}$;

$$\overline{g} = grad\, W = \begin{bmatrix} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \\ \frac{\partial W}{\partial z} \end{bmatrix}. \tag{2.15}$$

The components of gravity vector $\overline{g}$ are:

$$
\begin{aligned}
g_x &= \frac{\partial W}{\partial x} = \frac{\partial V}{\partial x} + \frac{\partial \Phi}{\partial x} = \frac{\partial V}{\partial x} + \omega^2 X \\
g_y &= \frac{\partial W}{\partial y} = \frac{\partial V}{\partial y} + \frac{\partial \Phi}{\partial y} = \frac{\partial V}{\partial y} + \omega^2 Y \ . \\
g_z &= \frac{\partial W}{\partial z} = \frac{\partial V}{\partial z} + \frac{\partial \Phi}{\partial z} = \frac{\partial V}{\partial z} + 0
\end{aligned}
\tag{2.16}
$$

The magnitude of gravity vector $\overline{g}$ is

$$
g = \sqrt{g_x^2 + g_y^2 + g_z^2}. \tag{2.17}
$$

The direction of gravity vector $\overline{g}$ is the direction along the plumb line. The surface with constant potential (W=Constant) is called equipotential surface. There are infinite equipotential surfaces. Equipotential surface that coincide with sea level without disturbance is called geoid.

2.3 Normal Gravity and Anomalous Gravity Field

Determination of normal gravity field is closely related to the definition of the reference ellipsoid (Ameti, 2006). A reference ellipsoid is an ellipsoid of revolution with its center at the geocenter and its mass equal to the earth's mass. One of the most useful reference ellipsoid is GRS80 (Geodetic Reference System 1980), which is defined by the following parameters:

Major semi-axis a = 6378137.0 m

Reciprocal of flattening $1/f$= 298.2572221

Angular velocity of the earth ω= 7292115 10^{-11} rad/s

Earth's gravitational constant GM = 3986005x10^8 m^3/s^2

With these four parameters it is possible to compute the normal potential U and normal gravity γ on or outside the surface of the reference ellipsoid (Ameti, 2006).

The normal gravity potential at the surface of the reference ellipsoid, being constant for a level ellipsoid, is given by (Moritz, 1980):

$$
U_0 = \frac{GM}{\varepsilon} \arctan e' + \frac{1}{3}\omega^2 a^2, \tag{2.18}
$$

where ε is linear eccentricity of the ellipsoid

$$
\varepsilon = \sqrt{a^2 - b^2}, \tag{2.19}
$$

b is minor semi-axis of the ellipsoid

$$
b = a\sqrt{1 - e^2}, \tag{2.20}
$$

e and $e^{'}$ is the first and second eccentricity of the ellipsoid respectively

$$e = 2f - f^2, \tag{2.21}$$

$$e^{'} = \frac{e}{\sqrt{1-e^2}}. \tag{2.22}$$

The normal gravity γ can be calculated on the surface of the ellipsoid by the closed formula of Somigliana (Heiskanen & Moritz, 1967)

$$\gamma = \gamma_e \frac{1 + k\sin^2\varphi}{\sqrt{1-e^2\sin^2\varphi}}, \tag{2.23}$$

where

$$k = \frac{b\gamma_P}{a\gamma_e} - 1, \tag{2.24}$$

$$\gamma_e = \frac{kM}{ab}\left(1 - \frac{3}{2}m - \frac{3}{14}e^{'2}m\right), \tag{2.25}$$

$$\gamma_P = \frac{kM}{a^2}\left(1 + m - \frac{3}{7}e^{'2}m\right). \tag{2.26}$$

a= major semi-axes of ellipsoid

b= minor semi-axes of ellipsoid

γ_e= normal gravity at the equator

γ_p= normal gravity at the poles

e^2= square of the first ellipsoidal eccentricity

φ= geodetic latitude

For the calculation of the normal gravity at the points outside the reference ellipsoid, the Taylor series expansion can be used for the upward continuation of the normal gravity from normal gravity on the surface of the ellipsoid to the point outside it. A frequently used Taylor series expansion for normal gravity above the ellipsoid with a positive direction downward along the geodetic normal to the reference ellipsoid is (Ameti, 2006):

$$\gamma_h = \gamma\left[1 - \frac{2}{a}(1 + f + m - 2f\ \sin^2\varphi)h + \frac{3}{a^2}h^2\right], \tag{2.27}$$

where

$$m = \frac{\omega^2 a^2 b}{GM}. \tag{2.28}$$

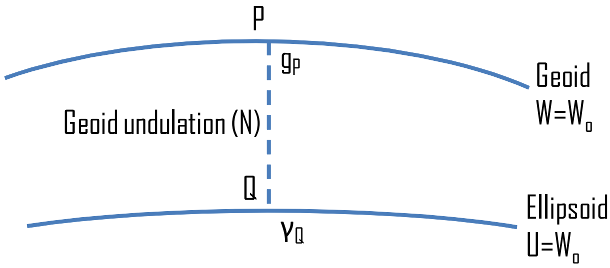

Figure 2.3: Geoid and reference ellipsoid

2.3.1 Disturbing Potential

Disturbing potential is defined as the difference between actual gravity potential W and the normal gravity potential U on the geoid (see Figure 2.3):

$$T_P = W_P - U_P. \tag{2.29}$$

2.3.2 Gravity Anomaly

Gravity anomaly is defined as the difference between actual gravity g on geoid and the normal gravity γ on the reference ellipsoid (see Figure 2.3):

$$\Delta g = g_P - \gamma_Q. \tag{2.30}$$

Equation (2.30) represents the difference of values in the magnitude between the gravity on the geoid and the normal gravity on the ellipsoid, while the difference between their directions is called deflection of vertical.

The relationship between disturbing potential T and gravity anomaly Δg results in (Heiskanen & Moritz, 1967):

$$\Delta g = -\frac{\partial T}{\partial n} + \frac{1}{\gamma}\frac{\partial \gamma}{\partial n}T. \tag{2.31}$$

This formula is called the fundamental equation of physical geodesy. It is a boundary value for the geodetic boundary value problem. In spherical approximation, this formula can be written as

$$\Delta g = -\frac{\partial T}{\partial n} - \frac{2}{R}T, \tag{2.32}$$

where R is the mean radius of the earth. Spherical approximation does not mean that a sphere replaces the reference ellipsoid for reference surface for the geoid. Spherical approximation only means that the errors of the order of the flattening are neglected.

2.3.3 Gravity Disturbance

Gravity disturbances is defined as the difference between actual gravity g and normal gravity γ on the geoid (see Figure 2.3):

$$\delta g_P = g_P - \gamma_P. \tag{2.33}$$

Disturbing potential T and gravity disturbances ∂g is related by

$$\delta g = -\frac{\partial T}{\partial n}, \tag{2.34}$$

where n denotes the ellipsoidal normal direction, or on spherical approximation

$$\delta g = -\frac{\partial T}{\partial r}. \tag{2.35}$$

2.3.4 Geoid Undulation

Geoid undulation N is defined as the distance between one point on the geoid and its projected point on the ellipsoid (see Figure 2.3). The geoid undulation is related to the disturbing potential T by Bruns' formula (Heiskanen & Moritz, 1967):

$$N = \frac{T_P}{\gamma_Q}. \tag{2.36}$$

2.4 Geodetic Boundary Value Problem

The geodetic boundary value problem is the determination of the earth's physical surface from the values of gravity and gravity potential given on on it (Heiskanen & Moritz, 1967). In modern geodesy, geodetic boundary value problems deal with the determination of gravity potential on or outside earth's surface from the data defined on or outside earth's surface. There are four approaches of geodetic boundary value problem. They are Stokes' approach, Hotine's approach, Molodensky's approach and Hotine-Molodensky's approach (see Figure 2.4). The formulation of the geodetic boundary value problem depends on the boundary surface chosen. The boundary surface can be a sphere, an ellipsoid, a telluroid, or even the earth's surface. In this paper only spherical boundary surface is discussed.

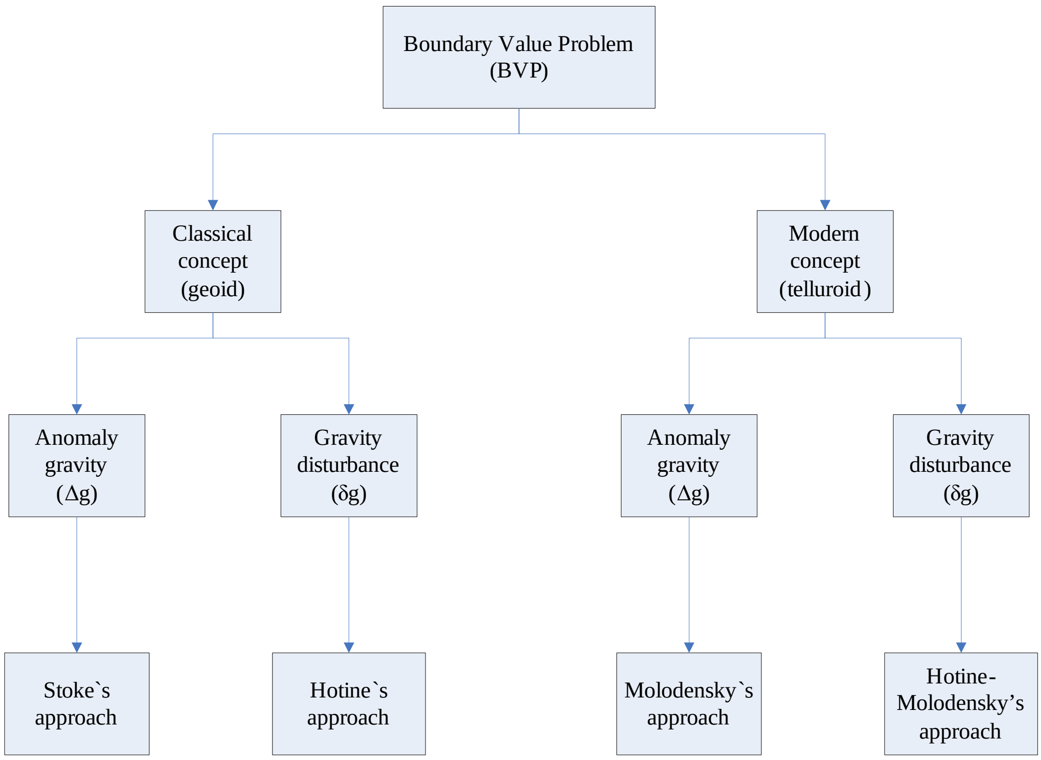

Figure 2.4: Four approaches of geodetic boundary value problem

2.4.1 Stokes' Approach of Geodetic Boundary Value Problem

The definition of the boundary surface used in Stokes' approach is the ellipsoid. The problem of Stokes may be formulated as: Determination of the geoid based on the gravity potential $W = W_0 = const$ and gravity g given at all points on the geoid. The gravity anomaly according to Stokes' approach is given in (2.30), disturbing potential in (2.29) and geoid undulation is expressed by Brun's formula as (2.36).The gravimetric solution of the geoid is based on gravity anomaly which is referred to the geoid, and the solution to the geodetic boundary value problem for spherical approximation can be represented by Stokes integral formula as (Heiskanen & Moritz, 1967)

$$T = \frac{R}{4\pi} \int\int_{\sigma} \Delta g \, S(\psi) \, d\sigma \tag{2.37}$$

with

$$S(\psi) = \frac{1}{\sin(\frac{\psi}{2})} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) \tag{2.38}$$

where

R= Mean earth's radius
Δg = Gravity anomaly
$S(\psi)$= Stokes' function (ψ-spherical distance)
σ = Integration area (all over earth's surface)

2.4.2 Hotine's Approach

As an alternative to gravity anomaly, the disturbing potential T will be related to the gravity disturbance δg by the integral formula as follow

$$T = \frac{R}{4\pi} \int\int_{\sigma} \delta g \, K(\psi) \, d\sigma \tag{2.39}$$

with

$$K(\psi) = \frac{1}{\sin(\frac{\psi}{2})} - ln\left(1 + \frac{1}{\sin(\frac{\psi}{2})}\right), \tag{2.40}$$

which is the modern equivalent of Stokes formula (Moritz, 2010). The geoid height can be then calculated by adding (2.39) into (2.36)

$$N = \frac{T}{\gamma}.$$

However, in Stokes' approach and Hotine's approach gravity g must be given on the geoid. This is of course cannot be carried out since measurements are carried out on and outside earth's surface. To downward the gravity given on earth's surface to the geoid, some reductions must be applied. These reductions contain some assumptions about densities distribution, which is violating the fact, and therefore can cause systematical errors in the computation (for more information see Heiskanen & Moritz, 1967).

2.4.3 Molodensky's Approach

The definition of the boundary surface used in Molodensky's approach is the earth's surface (see Figure 2.5). The problem of Molodenksy may be formulated as: Determination of the physical earth's surface based on the gravity potential W and gravity g given on it. The definition of gravity anomaly according to Molodensky's approach is

$$\Delta g = g_{topo} - \gamma_{tell}, \tag{2.41}$$

the definition of disturbing potential is

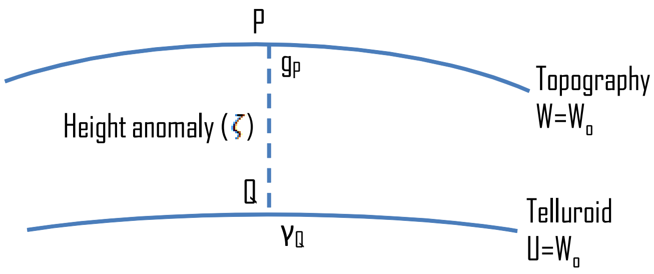

Figure 2.5: Topography and telluroid

$$T_{topo} = W_{topo} - U_{topo}, \tag{2.42}$$

and height anomaly is expressed by Bruns formula as

$$\zeta = \frac{T_{topo}}{\gamma_{tell}}. \tag{2.43}$$

They are closely related to (2.30), (2.29) and (2.36).

The gravimetric solution of the height anomaly is based on gravity anomaly which is referred to the earth's surface. Molodensky (1945) found an integral equation extended over the earth solved by a series whose first term is Stokes formula and the higher terms represent the topography (Moritz, 2010)

$$T = \frac{R}{4\pi} \int\int_{\sigma} \Delta g \, S(\psi) \, d\sigma + k_1 + k_2 + k_3 \ldots \, , \tag{2.44}$$

with $S(\psi)$ is Stokes function (see equation 2.38) and k_i is Molodensky correction (for more information see Heiskanen & Moritz, 1967).

2.4.4 Hotine-Molodensky's Approach

As an alternative to gravity anomaly, the disturbing potential T will be related to the gravity disturbance δg. Replacing gravity anomaly Δg with gravity disturbance δg, the disturbing potential T is given by

$$T = \frac{R}{4\pi} \int\int_{\sigma} \delta g \, K(\psi) \, d\sigma + k_1 + k_2 + k_3 \ldots \, , \tag{2.45}$$

with $K(\psi)$ is Hotine function (see equation 2.40) and k_i is Molodensky correction (for more information see Heiskanen & Moritz, 1967). The geoid height N can be then calculated by adding (2.44) into (2.43).